Abstract: This paper presents a simple but comprehensive solution of the Haus equation to describe the dynamics of mode-locking in a saturable absorber in colloidal nanoparticles. Soliton formation is considered as the main mechanism of pulse shaping. The results show a good matching with the experimental observations in which the suspensions have been introduced as suitable media for saturable absorption.

Keywords: Saturable Absorption, Soliton, Passive mode-locking, Colloidal nanoparticles
1 Introduction

Nanoparticles and Colloidal nanosuspensions are well-known for their high nonlinear response even at optically low intensities; their high performance of controlling light with light predicts the promising future of advances in all-optical devices and systems. Recently, Alexander Shamray and his group have experimentally shown that the optical bistability in colloidal solution of LaF$_3$:Er,Yb nanocrystals is truly attainable at low optical power[1]. In spite of multiple studies on nanosuspensions nonlinearity and Saturable Absorption (SA) via nanosuspensions [2-5], there seems to be a lack of theoretical model based on intracavity dominating equations to describe the dynamics of SA through these media.

2 Theoretical description of problem

Theoretical dynamics of pulse-shaping can be described by the Haus equation of mode-locking [6]:

$$ T_R \frac{\partial A(T,t)}{\partial T} = \left( -iD \frac{\partial^2 A}{\partial t^2} + ig \gamma A^2 + [g - l + D_s \frac{\partial^2}{\partial t^2} - q(T,t)]A(T,t) \right) A(T,t) $$

(1)

Where, $A(T,t)$ is slowly varying field envelop, $T$ is time scaled to one cavity round trip $T_R$, $t$ is time scaled to pulse duration, $D$ is Group Velocity Dispersion (GVD) and $D_s$ is gain dispersion; $\gamma = \frac{2\pi}{\lambda} n_2$ is the nonlinear coefficient required for Self Phase Modulatiom (SPM) whereas $\lambda$ is the center wavelength of the pulse, $n_2$ is the nonlinear refractive index of nanosuspension; $g$ and $l$ are the gain and loss of the cavity respectively; $q(T,t)$ is the response of the absorber. For simplicity, $g - l$ is substituted by the quantity $NET$ gain. It should be noted that intracavity loss $l$ is quite different from the loss due to absorption of the nonlinear media; the latter is postponed to be discussed. The corresponding pulse energy can be written as:

$$ E = \int^{\frac{\tau}{2}}_{-\frac{\tau}{2}} |A(T,t)|^2 dt $$

(2)

2.1 Soliton Regime

Soliton formation is considered as the dominant pulse-shaping mechanism. The function to describe the pulse shape is given below:

$$ f(t) = \sqrt{\frac{\tau}{\pi}} \text{sech}(t) $$

(3)

$\tau$ is pulse width which is defined as the full-width at half-maximum (FWHM) of the soliton. Soliton formation is a consequence of the balance between GVD and SPM inside the cavity; this in turn leads to the equation $\frac{D}{\tau^2} = \frac{\gamma E}{4\pi}$. The equation can be written as the pulse width related to the energy of the pulse:

$$ \tau = \frac{4|D|}{\gamma E} $$

(4)

On the other hand, energy rate equation is proved to be written as[6]:

$$ T_R \frac{\partial E}{\partial T} = 2g_{\text{NET}} - 2q(E) $$

(5)

Where:

$$ g_{\text{NET}} = \frac{g_{0,\text{NET}}}{1 + \frac{E}{E_s}} $$

(6)

Through the steady state, the ratio between intracavity GVD and gain dispersion is set to be at a certain value[6]. It is inevitable to introduce TPA (Two Photon Absorption) loss through the calculations in order to measure Minimum Pulse Energy (MPE) needed for the desired case of single pulse mode locking. Equation describing MPE is obtained according to [6]:

$$ E_{\text{min,SSA}} = \frac{q_m E_s}{\alpha_{\text{TPA}}} $$

(7)

Here, $q_m$ is the maximum absorber response, $\alpha_{\text{TPA}} \propto \frac{\alpha}{12D}$ is the loss pertained to TPA. The quantity $\alpha$ is the aforementioned absorption of nonlinear medium due to Rayleigh scattering.

2.2 Non-ideal gas nonlinearity model of nanosuspensions

According to this approach, the first order nonlinear refractive indices of $n_2$ is yielded as:

$$ n_2 = \frac{n_{2k}}{1 + 2(B_2/N_p f_0^2 + 3(B_3/N_p^3 f_0^2)} $$

(8)

In these equations, $n_{2k}$ is artificial Kerr nonlinearity, $B_2$ and $B_3$ are second and third
Virial coefficients, $V_p$ is the volume of nanoparticle, $f_{0}$ is the field free background filling factor of the colloidal nanoparticles. The first nonlinear loss of TPA due to Rayleigh scattering is given by 
$$\alpha = \frac{\sigma p_0}{I_c (1 + 2B_2/V_p f_{0} + 3B_4/V_p f_{0}^2)}$$
where $p_0$ is the initial nanoparticle density inside the colloid, $I_c$ pertains to the light intensity keeping the limit of exponential model for nonlinearity and 
$$\sigma = \frac{128\pi a^2 n_b^4}{3} \left( \frac{\alpha}{\lambda_0} \right)^2 \left( \frac{m^2 - 1}{m^2 + 2} \right)^2$$
is scattering cross section; $n_b$ is background refractive index of colloid and $m = n_p / n_b$ is the ratio of nanoparticles refractive index to $n_b$.

3 Analytical Results and Discussion

For a sinusoidal response of the absorber[8], one pulse operation is assumed. Equation (9) performs a simple but comprehensive analytical solution of the equation (5) to describe the dynamics of soliton regime inside mode-locked cavity for successive cavity round trips. Once the energy inside the cavity is increased, SA response will be strengthened and the intracavity energy will be diminished to its primary value. This implies that the field envelop inside the cavity is passively mode-locked.

$$E(T) = \frac{\frac{g_{0,NET}}{1 + \frac{E(T - T_k)}{E_s}} \times \alpha_{TPA}(1-q_M \cos(\pi E(T - T_k)))E(T - T_k)}{E_s}$$

The iterative form of equation (9) explicates the state of the system as a function of $q_M$ and $E_s$. For sufficiently small values of $q_M$ and $E_s$, cw stable operation will be provided if the gain of the medium is adequate. To ensure that high efficient laser cavity is attainable, $q_M$ should be increased; afterward, above a certain threshold value of $E_s$, the system transforms to mode-locking state. Figure 1 shows system transition from the primary cw state toward the chaotic state as $q_M$ and $E_s$ are increased.

3.1 Effect of Nanoparticle characteristics on the system state

Experimental measured data are taken from Majles Ara, et al.[3]. Calculations based on the presented model show that for the nanoparticles density order of 10 times larger than what is assumed, steady state mode-locking will be vanished and the system will experience a fully chaotic regime. Conversely, for 10 times lower order steady state mode-locking regime will no more be available.

3.2 MPE required for passively mode-locking

Equation (7) stands for MPE required for steady state mod-locking. Figure 2.a shows the dependence of MPE on the average diameter of nanoparticles. It can be seen that there is an optimum point in which the MPE is increased over/under the value. The procedure offers to use small sizes of nano-scaled particles. Indebted to the large values of $\alpha$ and $\gamma$ pertained to nanosuspensions, researchers believe/experience that passive mode-locking is attainable at extremely low powers [4-5]. Infinitesimal experimental reported values of $D$ inside nanosuspensions intensify this opinion (refer to equation (8)).
transforms to upper stable state), \( q_{M} = 0.3 \), c) Chaotic state (Inset shows how the system has instability over time), \( q_{M} = 0.6 \).

4 Conclusion

A simple, but comprehensive solution of Haus equation is presented. Using the Non-ideal gas model of nonlinearity for colloidal nanoparticles, a model is constructed to investigate Saturable Absorption inside nanosuspensions. These results seem to be truly matched with experimental observations.

![Graph](image)

**Figure 2:** MPE vs. nanoparticles size a) for assumed nanoparticles density b) for 10 times higher nanoparticles density

References


